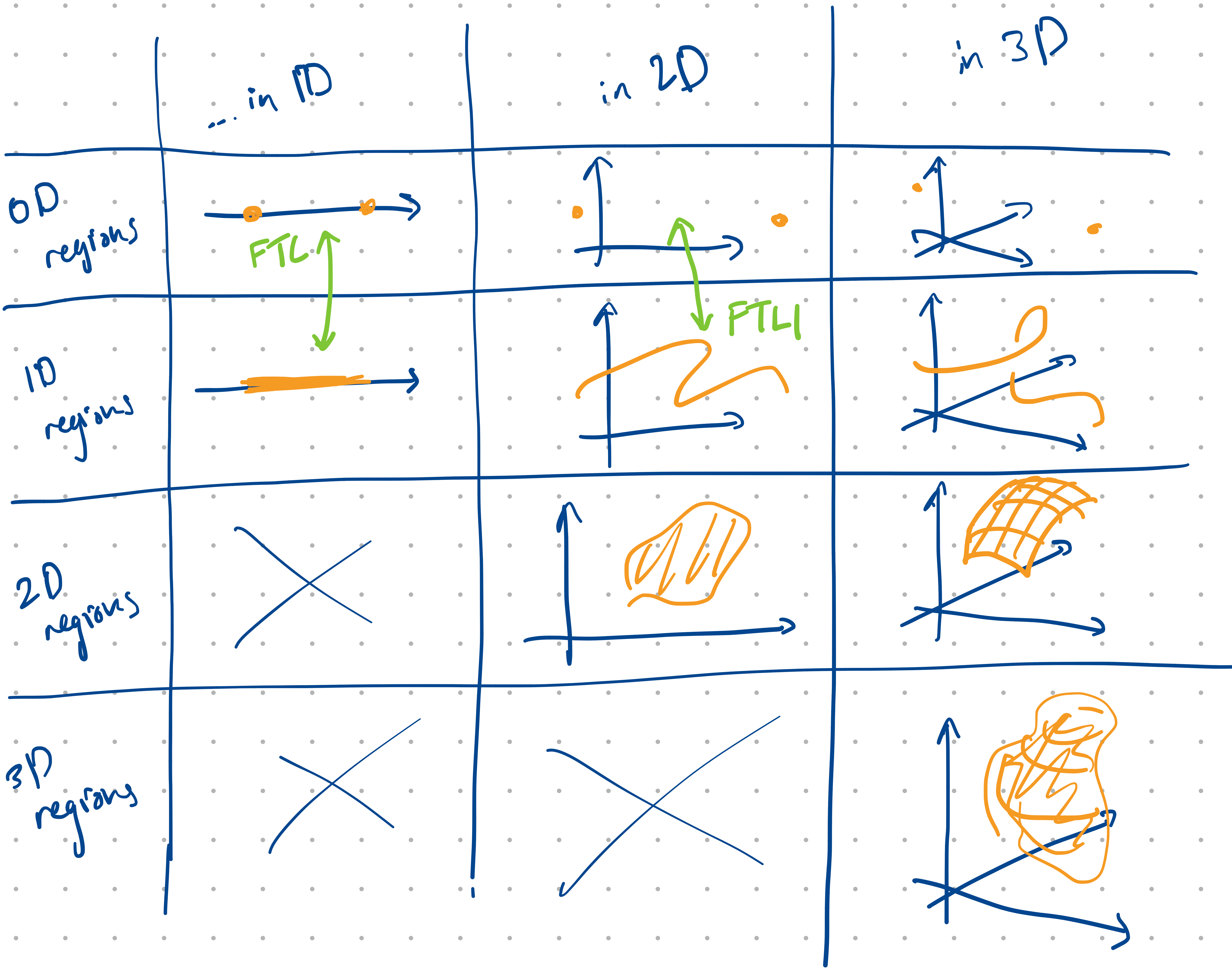


⚠ FTLI, "path-independence" etc...

are about <u>vector fields</u>, not scalar fields ($\mathbb{R}^2 \longrightarrow \mathbb{R}$)

i.e. they are about problems like

$$\int_C \vec{F} \cdot d\vec{r} \qquad \underline{\underline{\text{NOT}}} \qquad \int_C f \, ds$$

SVC: scalar functions $\xrightarrow{\text{differentiate}}$ scalar functions ①

e.g. $f(x) = x^2$ $\qquad$ $f'(x) = 2x$

MVC: scalar functions $\xrightarrow[\nabla]{\text{differentiate}}$ vector field ②

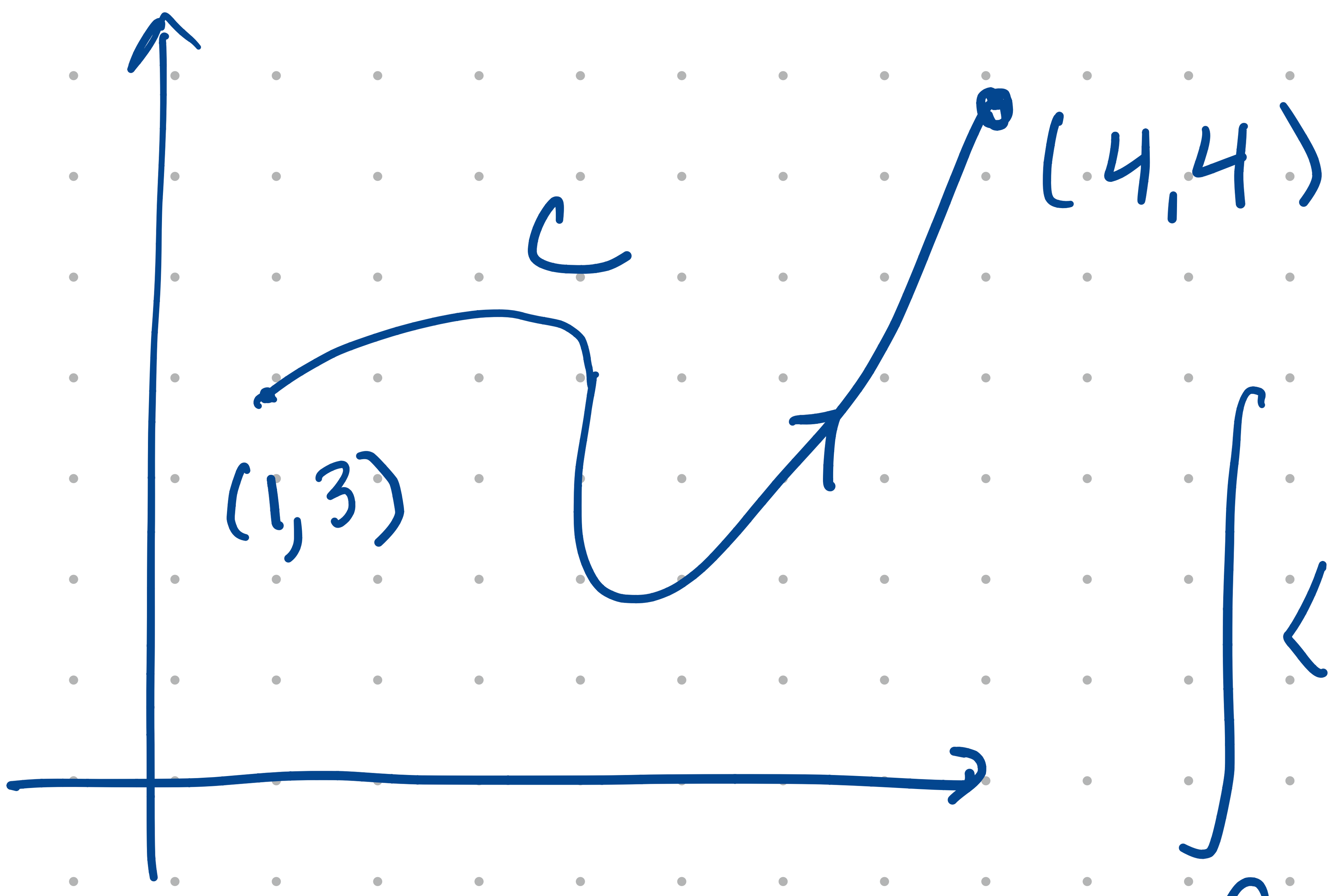
$f(x,y) = x^2 + y^2$ $\qquad$ $\nabla f(x,y) = \langle 2x, 2y \rangle$

⚠ Essential difference: Everything (reasonable) in ① comes from something on the left, but that's not the case in ②.

e.g. $\langle \underbrace{x^2}_{P}, \underbrace{y + x^2}_{Q} \rangle$.

$P_y \neq Q_x$ so this is not a gradient.

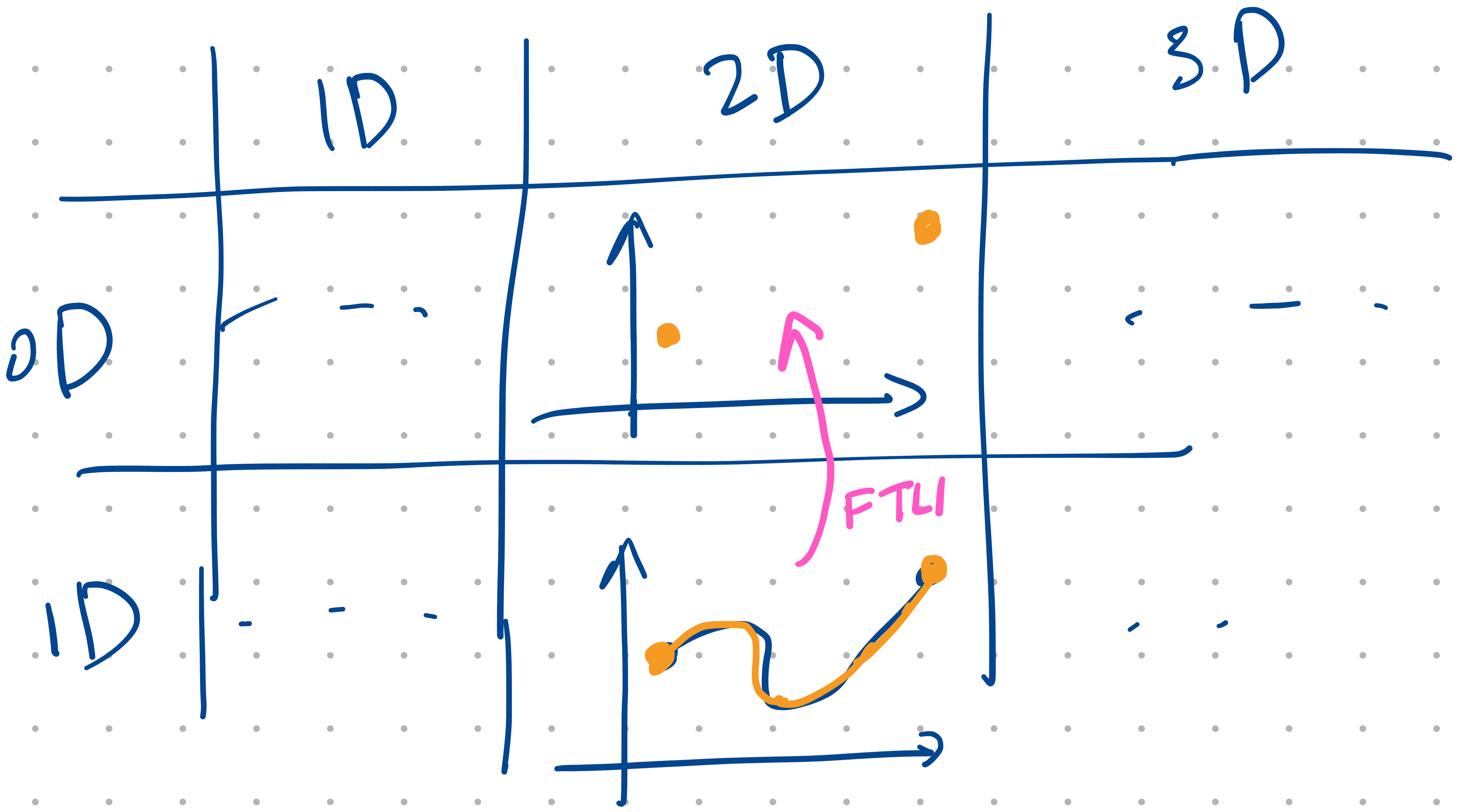
ex)



$$\int_C \langle 2x, 2y \rangle \cdot d\vec{r}$$

$$= \int_C \nabla (x^2+y^2) \cdot d\vec{r}$$

$$= (x^2+y^2)\Big|_{(x,y)=(1,3)}^{(4,4)} = 32 - 10 = \boxed{22}$$



$\vec{F}$ a vector field defined on some region R.

2

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$ for all paths C_1, C_2 in R that have the same endpoints

"path-independence"

1

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$ along every closed loop C in R

"conservative"

3

there exists a "potential function" f s.t. $\nabla f = \vec{F}$.

Clairaut's Thm.

if R is simply connected.

4

$$\nabla \times \vec{F} = \vec{0} \quad (\text{in 2D, } P_y = Q_x)$$

To show $\vec{F}$ <u>is</u> conservative:

③ could try to find a potential function.

④ if R is simply connected, then can use $P_y = Q_x$ to check.

To show $\vec{F}$ is <u>not</u> conservative:

③ show that integration results in a contradiction.

④ show $P_y \neq Q_x$ (recommended)

①,② Find a loop C such that $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

1) Look @ $(0,-5)$

$\leftarrow \qquad \rightarrow \qquad \downarrow \qquad \uparrow$

$\langle x, -y \rangle$ @ $(0,-5)$ is $\langle 0, 5 \rangle$

2) $\mathbb{R}^2$ is simply conn, and $P_y = 0 = Q_x$ so it is conservative.

Try to find $f(x,y)$ s.t. $\nabla f(x,y) = \langle x, -y \rangle$.

$f_x(x,y) = x$ so $f(x,y) = \frac{1}{2}x^2 + \underbrace{C(y)}_{\text{function of } y}$

$$-y = f_y(x,y) = 0 + C'(y)$$

thus $C(y) = -\frac{1}{2}y^2 + D$ constant.

e.g. $f(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + 42$.

3) @ $(0,-5)$ have $\longleftarrow$ so A. $\langle y,-x\rangle$

4) $P_y = 1 \neq -1 = Q_x$

so it is <u>not</u> conservative.

Alternative:

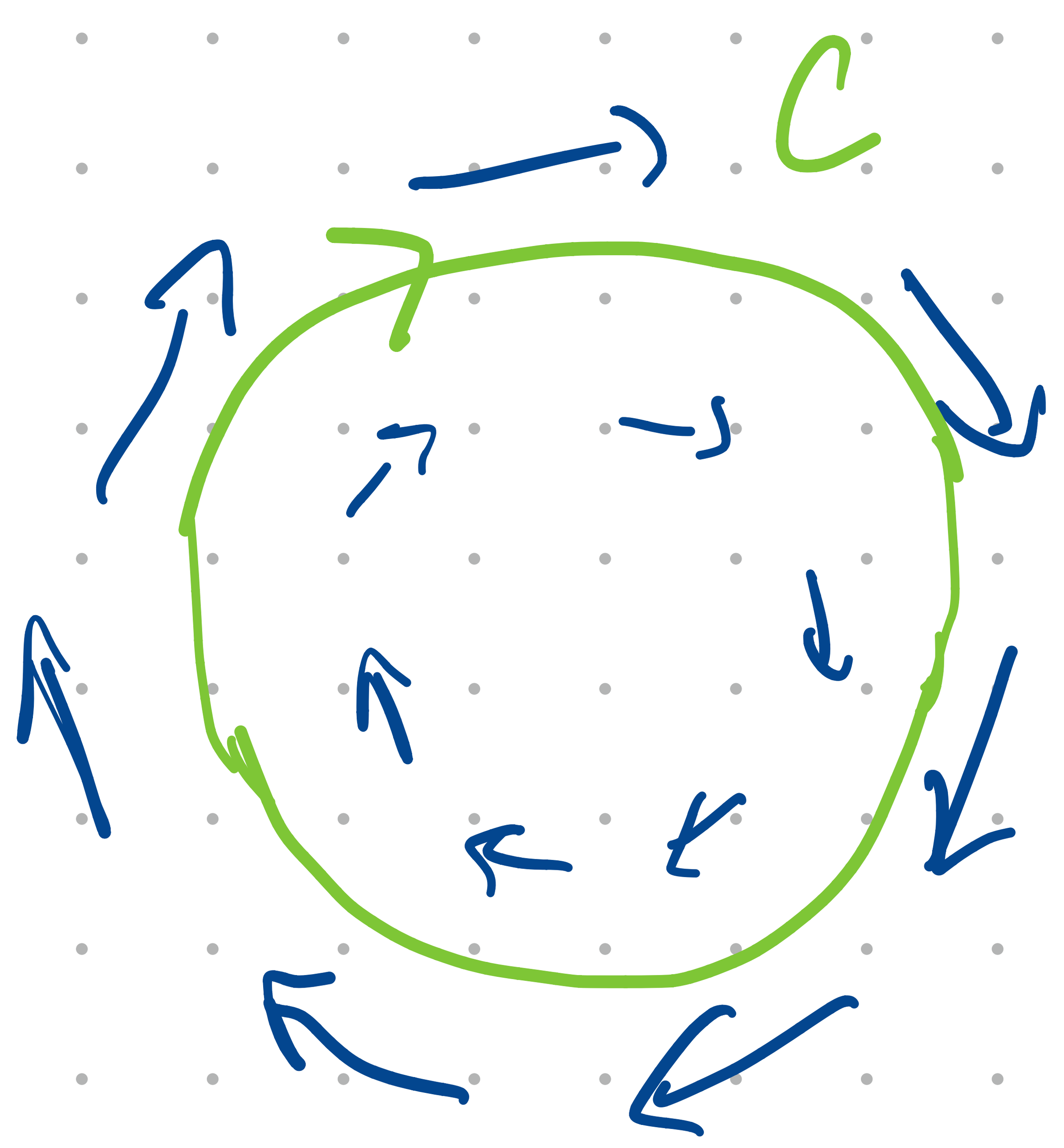
$f_x(x,y) = y$ so $f(x,y) = xy + C(y)$

$-x = f_y(x,y) = x + C'(y)$

<u>impossible</u> to have $C'(y) = -2x$.

no x here!!

Alternative:

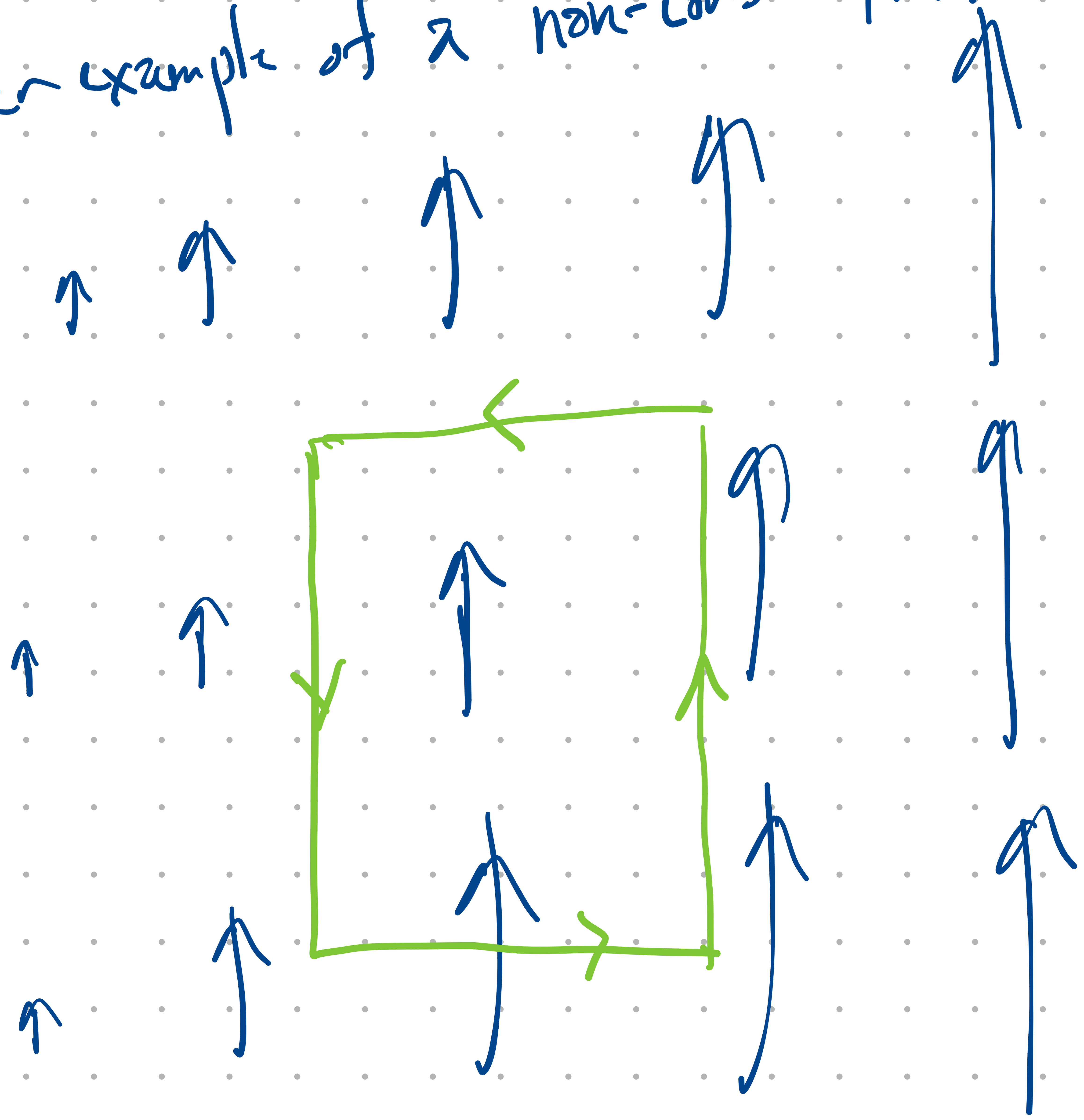


$C: \quad \vec{r}(t) = \langle \sin t, \cos t \rangle \quad 0 \le t \le 2\pi$

$$\int_C \langle y, -x \rangle \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left((\cos t)^2 + (\sin t)^2 \right) dt = 2\pi \neq 0.$$

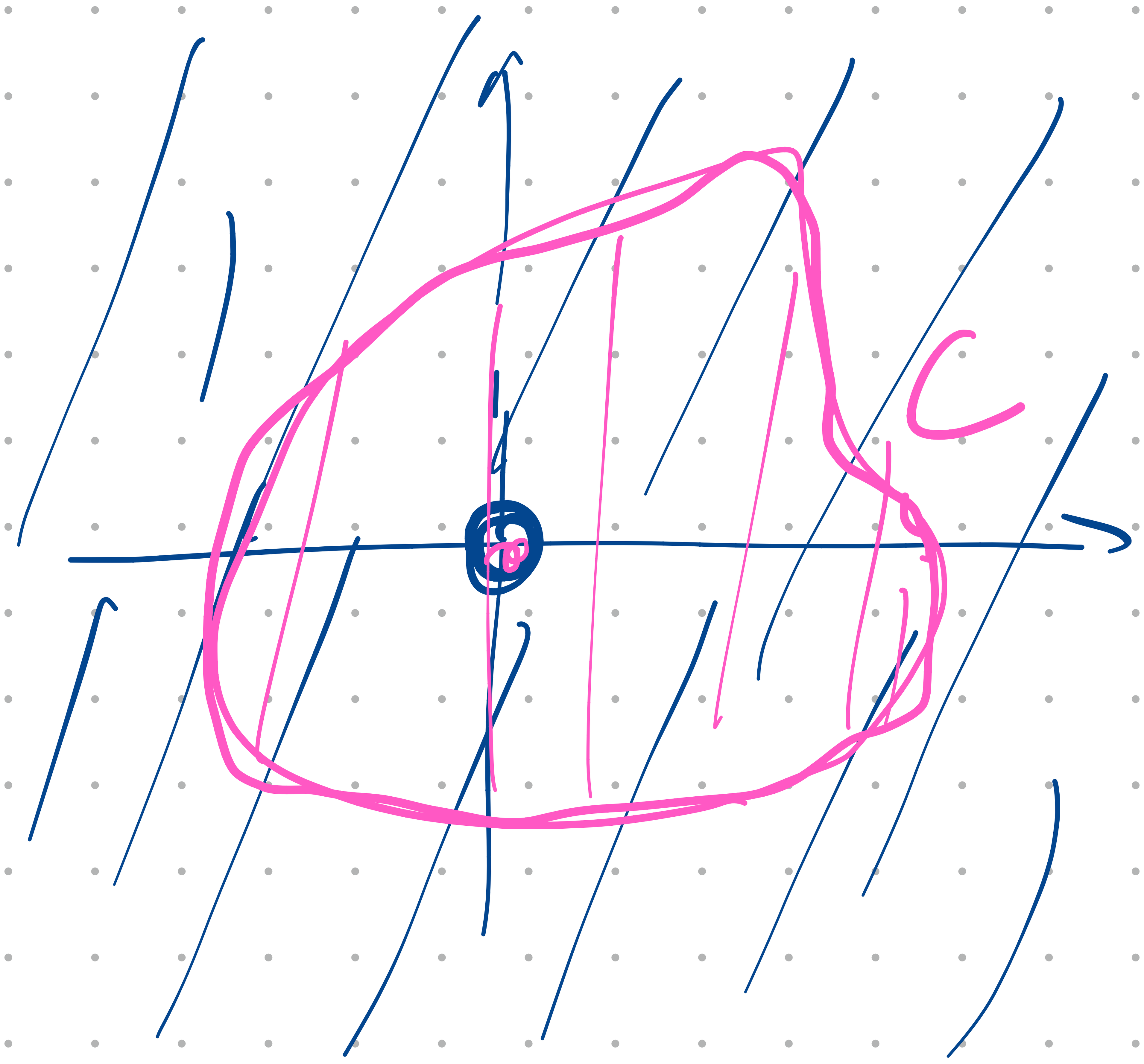
Another example of a non-cons. field.



5) In this case, $Q_x = P_y$.

However:

Domain of $\vec{F}$ is $\mathbb{R}^2 - 0$.



<u>Defn</u>. R (region) is simply connected if every loop in R encloses points only in R.

e.g. C is a loop in $\mathbb{R}^2 - (0,0)$ but it encloses $(0,0)$ which is not in $\mathbb{R}^2 - (0,0)$, so that region is <u>not simply connected</u>.

<u>Exercise</u>: Find a loop C s.t. $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

for the vec. field in 5.